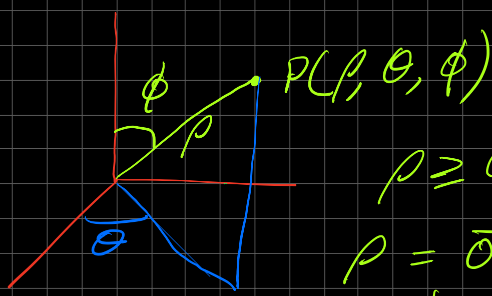
Triple integrals
(spherical coordinates)

Spherical coordinates



$\rho \geq 0$

$\rho = \overrightarrow{OP}$

$0 \leq \phi \leq \pi$

ϕ is the angle between the positive z-axis & the $\overrightarrow{OP}$.

θ is the same angle as in the cylindrical coordinates.

$z = \rho \cos\phi \qquad r = \rho \sin\phi$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$\rho^2 = x^2 + y^2 + z^2$

Ex) $(2, \frac{\pi}{4}, \frac{\pi}{3})$ spherical

$\longrightarrow$ cartesian

$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = \sqrt{3/2}$

$y = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = \sqrt{3/2}$ $\qquad \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

$z = \rho \cos\phi = 2 \cos\frac{\pi}{3} = 1$

$(\sqrt{3/2}, \sqrt{3/2}, 1)$

Ex) $(0, 2\sqrt{3}, -2) \longrightarrow$ spherical

$$\rho = \sqrt{0+12+4} = 4$$

$$\cos\phi = \frac{z}{\rho} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$

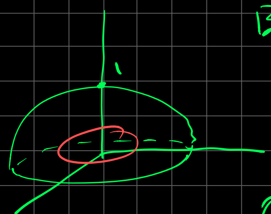
$\theta \neq \frac{3\pi}{2}$ because $y = 2\sqrt{3} > 0$

$$\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$$

$$\iiint_E f \, dV = \iiint f(\sim)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Ex) $\displaystyle\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$

$$B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$$



$B = \{(\rho, \theta, \phi)$

$$0 \le \rho \le 1$$

$$0 \le \theta \le 2\pi$$

$$0 \le \phi \le \pi$$

$$\int^{\pi} \int^{2\pi} \int^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

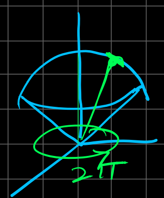
$$0 \quad 0 \quad \Big|_0$$

$$= \frac{4}{3}\pi(e-1)$$

Ex) Use spherical coordinates to find the volume of the solid that lies above

cone $z = \sqrt{x^2+y^2}$ &

below the sphere $x^2+y^2+z^2 = z$



Ans: The sphere passes through the origin & has center

$(0, 0, \frac{1}{2})$

$$z = \rho\cos\phi$$

$$\rho^2 = \rho\cos\theta \rightarrow \rho = \cos\theta$$

$$\rho^2 = x^2+y^2+z^2$$

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \sqrt{x^2+y^2}$$

$$\rho\cos\phi = \sqrt{x^2+y^2}$$

$$= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$

$$= \rho\sin\phi$$

$$\rho\cos\phi = \rho\sin\phi$$

$$\cos\phi = \sin\phi$$

$$\phi = \frac{\pi}{4}$$

$$0 \le \theta \le 2\pi$$

$$0 \le \phi \le \frac{\pi}{4}$$

$$0 \le \rho \le \cos\theta$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int^{\cos\theta} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{\pi}{8} \quad ?$$